Multiple representations in mathematical problem solving: Exploring sex differences

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Introduction

In mathematics education, visual representations are important both as an aid supporting reflection and as a means for the communication of mathematical ideas. Therefore, researchers believe that visual representations are an important aid for learning and problem solving. This study aims to shed light on the influence of two types of representations on additive problems and its gender variation. Specifically, we investigate the role of informational pictures and the use of the number line in contrast to the use of plain verbal description for the solution of one-step addition and subtraction problems. Sex differences in students' performance on the different modes of representation are also explored. Below we first discuss the nature and possible effects of different types of representation of arithmetic problems and then discuss the different structures in which these problems may be presented

Representations in Mathematics Learning

A representation is any configuration of characters, images or concrete objects that stands for something else. Schnotz (2002) suggests that text and visual displays belong to different classes of representations, namely descriptive and depictive representations respectively.

Given that a representation cannot describe fully a mathematical construct and that each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2002).

In elementary mathematics the number line is a representation that is widely used for the teaching of basic whole number operations and arithmetic in general. Gagatsis, Shiakalli and Panaoura (2003) consider the number line as a geometrical model, which involves a continuous interchange between a geometrical and an arithmetic representation. The simultaneous presence of the geometric and the arithmetic conceptualisation of number may limit the effectiveness of the number line and thus hinder the performance of students in arithmetical tasks (Gagatsis et al., 2003).

Structures of Addition and Subtraction Problems

Previous studies on one-step additive problems have identified three main types of semantic structures: change or transformation of a measure, combine or composition of two measures and compare two measures to each other (Nesher, Greeno, & Riley, 1982). In the present study we focused on one class of problems: one-step change (measure-transformation-measure) problems, which describe a transformation or a change (an increase or a decrease) in a starting situation to result in a final situation.

Varying the unknown and the type of the relationship in the problem of a specific semantic category generates different situations. Therefore, change problems include a total of six situations, distinguished by whether the problem describes a join (positive transformation) or a separate situation (negative transformation) and by the placement of the unknown. The unknown may be one of the three amounts (starting, transformation or final one) (Nesher et al., 1982; Christou & Philippou, 1998).

In the present study, we explore the use of two other modes of representation in addition to the verbal description. Specifically, we examine the role of informational pictures and the number line, on additive problem solving. Informational pictures provide information that is essential for the solution of the problem, because the content of the problem is based on the picture. The number line is a special kind of picture, because it may stand for an internal arrangement of the relations between numbers, which can be contrasted to the direction provided by informational pictures.

Research Questions of the Study

This study aims to contribute to our understanding of the role of the different types of representations discussed, of their possible interactions with the mathematical structure of the problems and of the sex differences in students' solutions of additive problem solving with multiple representations.

The research questions are the following:

1. What are the effects of different types of representation on the solution of additive mathematical problems?
2. What are the interactions of the different types of representations with the mathematical structure and more specifically with the placement of the unknown in the problems?

3. How do performance and the structure of the processes involved in the solution of additive mathematical problems vary between boys and girls?

Method

Participants

A total of 1491 primary school students (776 boys and 715 girls) randomly selected from urban and rural schools in Cyprus were examined. The sample involved students from 6 to 9 years of age from three grades. Genders were about equally represented in each grade. In particular, first grade students were comprised of 252 boys and 249 girls; second grade students were comprised of 243 boys and 243 girls; and third grade students were comprised of 281 boys and 223 girls.

Tasks

The test included 20 one-step change problems of additive structures (a+b=c). These problems differed in terms of the following three dimensions: (a) the direction of the transformation or the type of the relation, (b) the placement of the unknown element in the structure and (c) the mode of representation (Table 1).

Table 1

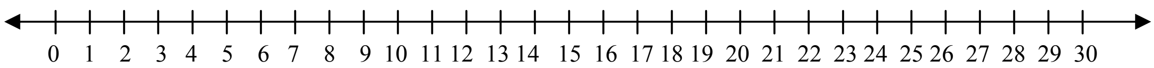
Specification table of the problems included in the test

	Join situation (b>0)			**Separate situation (b<0)**		
	Placement of the unknown			**Placement of the unknown**		
Type of representation	***Initial amount (α)***	***Transformation (b)***	***Final amount(c)***	***Initial amount (α)***	***Transformation (b)***	***Final amount(c)***
Text (verbal description)	VJa*	VJb	VJc	VSa	VSb	VSc
Text with informational picture	IJa	IJb	IJc	ISa	ISb	ISc
Text with number line	LJa	LJb	LJc	LSa	LSb	LSc

*Explanation of the symbolization: Symbols V, I and L at the first position stand for the mode of representation of the problem: V→verbal (written text), I→text with informational picture and L→text with number line; symbols J and S at the second position stand for the mathematical relationship of the problem: J→ join situation and S→separate situation; symbols a, b and c at the third position represent the placement of the unknown: a→ initial amount, b → transformation and c → final amount.

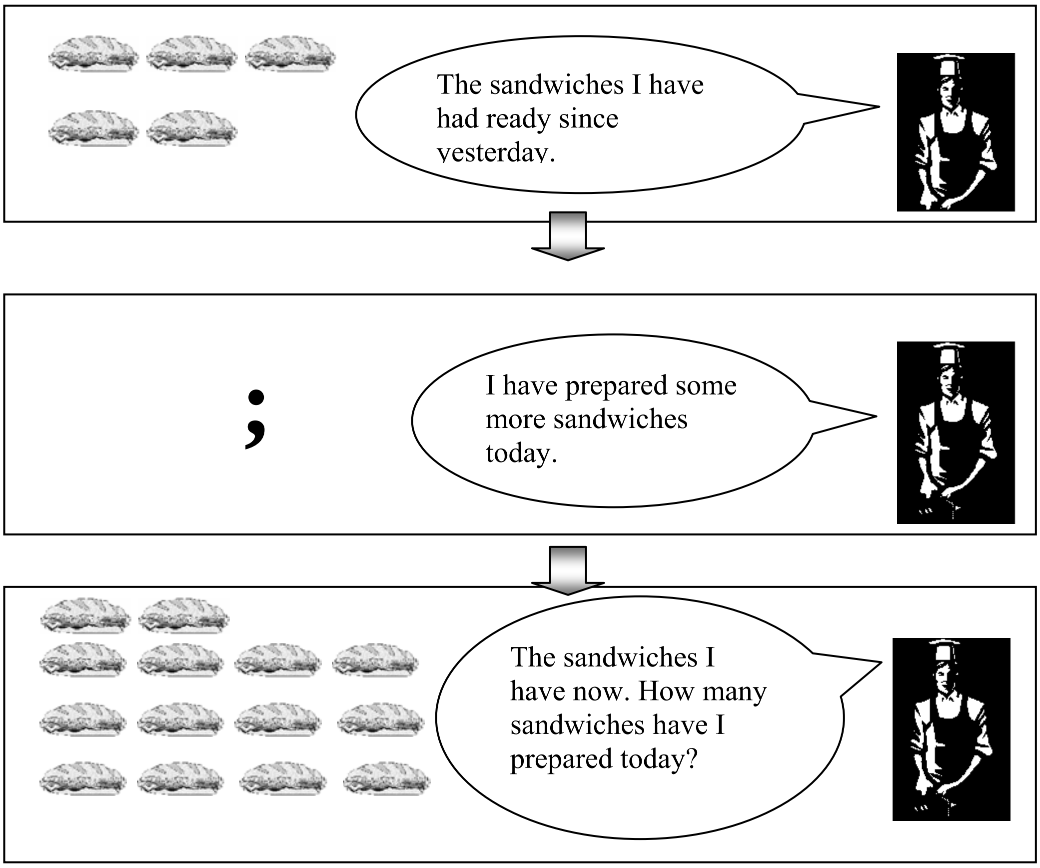
Examples of the tasks

1. I had 8 pounds before my birthday. I received some more money on my birthday. Now I have 15 pounds. How much money did I receive on my birthday? (Problem with number line-LJb)



2. Yesterday John had 7 pencils in his pencil case. He bought some more pencils and now he has 15 altogether. How many pencils did he buy? (Verbal problem-VJb)

3. Problem represented as an informational picture (IJb)



Results

Effects of Representations on Problem Solving Performance

Multivariate analysis of variance (MANOVA) has shown that the main effect of representation was strong {$F(2,1472)=194.280$, $p<0.0001$, $\eta^2=0.209$}. Performance on verbal problems ($\overline{X}$ = 1.561), was higher than performance on problems with number line ($\overline{X}$ = 1.480), which in turn was higher than performance on problems involving informational pictures ($\overline{X}$ = 1.351). These findings suggest that students found the problems involving the informational picture more difficult than the verbal problems and the verbal problems with number line. In most problem types the presence of the number line exerted little, but significant effect. In summary these findings indicate that the mode of representation influences problem solving. However, it is suggested that representations (such as informational pictures or number line) do not always facilitate problem solving or they may even hinder it because they need more complex mental processes relatively to other modes of representation.

Structure

Confirmatory factor analysis was used to explore to explore the structural organization underlying students' processes in the solution of additive problems. A Structural Equation Modeling computer program, namely MPlus, was used for the analysis (Muthén & Muthén, 2004). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: The observed values for X^2/df should be less than 2.5, the values for CFI (Comparative Fit Index) should be higher than .9 and the RMSEA (Root Mean Square Error of Approximation) values should be lower than .06.

The model that was examined integrates the effects of the mode of representation and the placement of the unknown on additive problem solving. Its structure was based on the major hypothesis of the study which stated that there is an important interaction of the type of representation and the placement of the unknown in additive problem solving. This model consists of four first-order factors: verbal descriptions of complex mathematical structures (Verbal, unk.ab), informational pictures of complex structures (Picture, unk.ab), verbal descriptions of complex structures accompanied by the number line (Number line, unk. ab) and situations in all of the three representations with the unknown at the final amount (unk. c)]. These factors were measured by the tasks involving the corresponding representation and placement of the unknown. The second-order factor stands for the general additive problem solving ability. The model was found to fit the data reasonably well according to the fit indices [X^2(131)=544.716, CFI= .965, RMSEA=.046].

The common first-order factor corresponding to the solution of the problems with the unknown in the final amount in all the three representations indicates that students solved the simple problems by activating similar processes across the different representations. The other three first-order factors standing for the solution of the problems with the unknown in the transformation and in the initial amount in the three representations, separately, suggest that students employed different processes across the different representations in solving complex problems. This structure suggests that multiple representations had a stronger impact on the solution of problems of a complex mathematical structure rather than on the solution of problems of a simple structure.

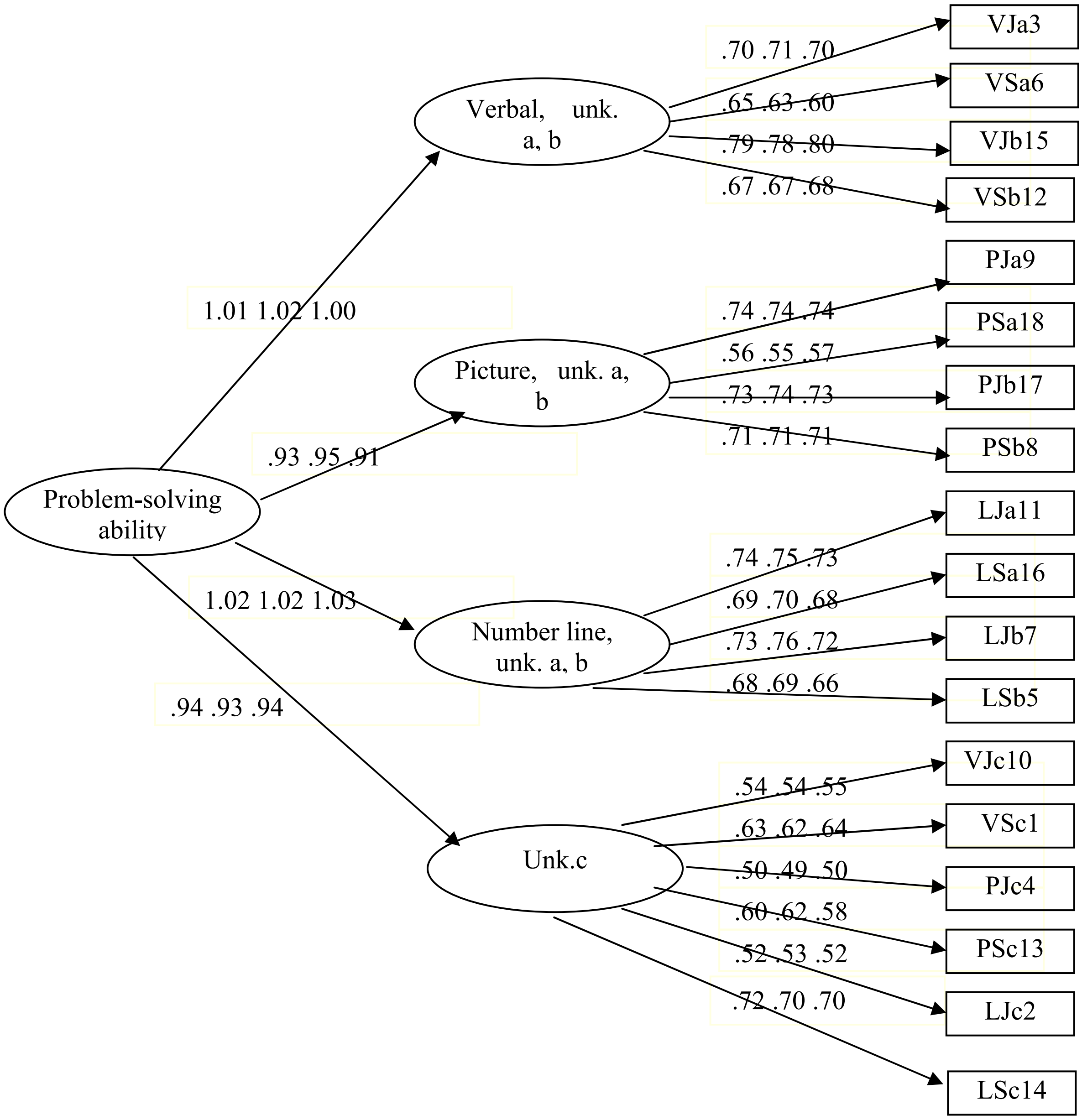


Figure 1: The confirmatory factor analysis model accounting for performance on the test by the whole sample, boys and girls, separately.

Note: The first, second and third coefficient of each factor stand for the application of the model on the performance of the whole sample, boys and girls, respectively.

Sex differences

The results of multivariate analysis of variance (MANOVA) showed that the effect exerted by sex was not significant $\{F\ (1,1473) = .588,\ p=.443,\ \eta^2=.000\}$ on students' additive problem solving performance. Boys ($\overline{X}$ =1.499) and girls ($\overline{X}$ = 1.481) performed equally well. Similar results were obtained in each grade separately as illustrated in Table 2. However, despite the statistically insignificant differences, it is noteworthy that in grade 1, boys slightly outperformed girls, in grade 2, boys' performance was almost identical to girls' performance, while in grade 3 girls slightly outperformed boys.

Table 2

Girls' and boys' performance by grade

Group	*Grade 1*	*Grade 2*	*Grade 3*
Girls	*1.096*	*1.595*	*1.751*
Boys	*1.180*	*1.597*	*1.720*

The interaction of gender with the mode of representation was not significant [$F_{(2,1472)}$=1.333, p=.264, η^2 =.002], indicating that the differences between boys and girls did not vary with representational type. Table 3 shows that boys and girls exhibited similar problem solving performance in each type of representation. They both encountered greater difficulty in the solution of problems represented as informational pictures compared to the other types of problems.

Table 3

Girls' and boys' performance by representational type

Group	*Verbal problems*	*Picture problems*	*Number line problems*
Girls	*1.569*	*1.379*	*1.495*
Boys	*1.602*	*1.379*	*1.516*

To test for possible differences between boys and girls in the structure of the model described above, multiple-group analysis was applied, where the CFA model was fitted separately on boys and girls. The model was consistent with the data [X^2 (276) =741.621; CFI=.961; RMSEA=.048], providing support to the invariance of this structure between boys and girls. The findings revealed that both boys and girls dealt flexibly and similarly with problems of a simple structure regardless of the mode of representation. However, when they confronted problems of a complex structure they activated distinct cognitive processes in their solutions with reference to the mode of representation. These results suggest that apart from the structure of the problem, the different modes of representation do have an effect on additive problem solving. There is an important interaction between the mathematical structure and the mode of representation in problem solving.

In an attempt to examine whether the particular structure remains invariant between boys and girls with age development, the above model was tested on girls and boys in each grade separately. The model had a good fit to the data of first grade's boys and girls [$X^2(276)=449.815$, CFI=0.942, RMSEA=0.050] and a barely acceptable fit to the data of second grade's boys and girls [$X^2(276)=519.138$, CFI=0.920, RMSEA=0.060], separately. Therefore, the same structure holds for boys and girls of these two age groups.

This was not the case in third grade, as the fit of the model on boys and girls of the particular grade was poor [$X^2(276)=658.382$, CFI=0.877, RMSEA=0.074]. The model seemed to apply to the boys of the particular grade (after some minor modifications), but not to the girls. This suggests that the particular structure was not sufficient to describe the solution of the additive problems by third grade girls. The application of the model in third grade students as a whole, was acceptable [$X^2(131)=334.744$, CFI=0.931, RMSEA=0.056], but the relations among the abilities involved (factor loadings) were weaker compared to the younger students'. This indicates that the dependence of the older students' solution processes on the mode of representation and the placement of the unknown was not as strong as the younger students' dependence. The particular discrepancy may be due primarily to third grade girls' problem solving behaviour. The fact that the only grade at which girls outperformed boys was the third grade suggests that the handling of representations and mathematical structure of the problems starts to become more flexible and efficient among the girls, rather than the boys, of the particular age.

Discussion

The results provided a strong case for the role of different modes of representation in combination with the placement of the unknown in additive problem solving. Informational pictures may have a rather complex role in problem solving compared to the use of the other modes of representation. The weak performance on these problems may have been caused by the fact that the very interpretation of the informational picture requires extra and perhaps more complex mental processes relative to the verbal mode of representation. That is, the thinker needs to draw information from different sources of representation and connect them. Boys and girls in the whole sample and in each grade exhibited similar levels of performance both in general and at each representational type of problems.

Boys and girls in first and second grade made sense of additive problems in multiple representations by using similar processes. This phenomenon was stronger among first grade students. Third grade boys and girls, despite their similar performance, were found to activate different processes in problem solving with multiple representations. Third graders used processes that were less dependent on the mode of representation and thus on its interaction with the placement of the unknown compared to younger students. Older students could be able to recognize the common mathematical structure not only of the simple problems, but also of the complex problems in different representations and deal more flexibly with them than younger students. This assertion is in line with Gagatsis and Elia's (2004) finding that development generates general problem-solving strategies that are increasingly independent of representational facilitators. This study indicates that girls probably begin to develop or employ explicitly and systematically these strategies earlier than boys.

It would be theoretically interesting and practically useful if this inference was further examined in a future study. This would require a longitudinal study combining quantitative and qualitative approaches to map the process activated by boys and girls at different stages of the particular age span.

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